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NAVIGATIONAL TECHNIQUES FOR INTERSTELLAR RELATIVISTIC FLIGHT

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13. ABSTRACT

In this study closed form expressions are developed for the inertial position and velocity of a spacecraft traveling at relativistic velocities in terms of parameters which can be measured by an observer located on the spacecraft. It is assumed that (1) the guidance system maintains a one-dimensional direction of travel toward the destination star, (2) there are no gravitational or drag forces acting on the spacecraft, and (3) the stars used for measurement are located in the XY plane. Using the Theory of Relativity transformation equations between an event in the sun centered XYZ coordinate system and the same event in the spacecraft centered xyz coordinate system are used to develop a wavelength shift equation, which relates the inertial wavelength λ of light emitted from a star to the apparent wavelength λ' measured by an observer on the spacecraft, and a relativistic aberration equation which relates the inertial angle θ of the position of a star to the apparent position angle θ' measured by an observer on the spacecraft. The wavelength shift equation is used to develop the velocity measuring method and the relativistic aberration equation is used to develop the position measuring method. The methods are shown to be identical for spacecraft traveling at either constant velocity or constant acceleration. An error analysis shows that the error in velocity determination is minimized by measuring the wavelength and position angle of light emitted by the sun and the error in position determination is minimized by measuring the position angle of stars for which the light emitted does not appear to shift in wavelength.

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NAVIGATIONAL TECHNIQUES FOR INTERSTELLAR RELATIVISTIC FLIGHT

THESIS

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Captain USAF

GA/MC/71-2

NAVIGATIONAL TECHNIQUES FOR INTERSTELLAR

RELATIVISTIC FLIGHT

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

David K. McMaster, B.S.M.E.

Captain USAF

Graduate Astronautics

June 1971

Preface

Man has always attempted to explore the unknown regions of his surroundings. The latest region which man has begun to explore is space. Some people seem to believe that man's exploration of space ended when the first man set foot on the moon. I believe that the exploration of our solar system and the universe has just begun. As long as men have gazed at the stars some men have dreamed of traveling to them.

This thesis is a preliminary study into the problem of navigating a spacecraft to the stars. Originally the goal was to develop a complete navigation system, however it became necessary to limit the study to determining the position and velocity of the spacecraft. Closed form expressions for spacecraft position and velocity, during either constant velocity or constant acceleration flight, are developed using the theory of relativity. These expressions are used to develop methods of measuring spacecraft position and velocity, using instruments located on-board the spacecraft, and are analyzed to determine how to minimize the error in making the measurements.

I wish to express my appreciation to my thesis advisor Professor G. M. Anderson, for suggesting the study and providing timely advice and encouragement as the study progressed and to Mrs. Imogene J. Hoffer for her patient and expert typing of the thesis. I also wish to express a sincere and special word of thanks to my wife, Sue, and our children, Dean, Patti Sue, and Mark, for their patience and loving understanding during the months of work on the thesis.

David K. McMaster

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List of Symbols

b	Speed of a plane wave
c	Speed of light in the inertial coordinate system
f	Frequency of an electromagnetic wave
g	Acceleration measured on the spacecraft
P	Phase of a plane wave
T	Time measured in the inertial coordinate system
t	Time measured on the spacecraft
V	Velocity measured in the inertial coordinate system
w	Speed of light in the spacecraft centered coordinate system
X,Y,Z	Axes of the inertial coordinate system
x,y,z	Axes of the spacecraft centered coordinate system
α	Angle between a light ray and the X axis
γ	Angle between the normal to a plane wave and the X axis
θ	Angle between the line of sight to a star and the X axis
λ	Wavelength of an electromagnetic wave

Subscripts

d	Destination star
e	Earth
s	Star
v	Spacecraft
x	Component of a vector quantity in the x direction
y	Component of a vector quantity in the y direction

Superscripts

$'$	Quantity measured in the spacecraft centered coordinate system

Abstract

In this study closed form expressions are developed for the inertial position and velocity of a spacecraft traveling at relativistic velocities in terms of parameters which can be measured by an observer located on the spacecraft. It is assumed that (1) the guidance system maintains a one-dimensional direction of travel toward the destination star, (2) there are no gravitational or drag forces acting on the spacecraft, and (3) the stars used for measurement are located in the XY plane. Using the Theory of Relativity transformation equations between an event in the sun centered XYZ coordinate system and the same event in the spacecraft centered xyz coordinate system are used to develop a wavelength shift equation, which relates the inertial wavelength λ of light emitted from a star to the apparent wavelength λ' measured by an observer on the spacecraft, and a relativistic aberration equation which relates the inertial angle θ of the position of a star to the apparent position angle θ' measured by an observer on the spacecraft. The wavelength shift equation is used to develop the velocity measuring method and the relativistic aberration equation is used to develop the position measuring method. The methods are shown to be identical for spacecraft traveling at either constant velocity or constant acceleration. An error analysis shows that the error in velocity determination is minimized by measuring the wavelength and position angle of light emitted by the sun and the error in position determination is minimized by measuring the position angle of stars for which the light emitted does not appear to shift in wavelength.

NAVIGATIONAL TECHNIQUES FOR INTERSTELLAR RELATIVISTIC FLIGHT

I. Introduction

Background

Manned and unmanned exploration of the moon and other planets in our solar system has already passed from the realm of science fiction to reality. As exploration of the planets in our solar system continues, the possibility of attempting to reach the nearest stars in our galaxy will undoubtably be given serious consideration as a natural extension of the space exploration program. Due to the long distances involved in attempting to reach other stars, interstellar spaceflight will be improbable unless spacecraft capable of traveling at relativistic speeds are developed. To reach the stars, navigational techniques must be developed to guide these spacecraft traveling at relativistic velocities.

There have been many theoretical investigations conducted to develop the propulsion systems needed to attain relativistic velocities. Much research has been accomplished on ion engines and working models have been constructed and tested at non-relativistic speeds. The ideal photon engine and both nuclear fission and nuclear fusion devices have been proposed as propulsion systems for relativistic rockets. Other studies have shown that relativistic propulsion systems are theoretically possible for interstellar flight (Refs 1,2,3),

awaiting only the necessary technological breakthrough to become reality.

Many researchers have predicted the visual effects which would be observed when traveling at relativistic speeds (Refs 8,9,11). These effects include aberration (position shift) due to the velocity of the observer, relativistic Doppler frequency shifts, and time dilatation (the slowing of clocks) as predicted by the theory of relativity. These effects, when combined, will produce severe distortions in the appearance of the stellar constellations to an observer on board the spacecraft which travels at relativistic speeds. As an example, at very high speeds (0.99 times the speed of light) the entire visible universe would appear to be contracted into a ring extending from 23° to 35° ahead of the spacecraft (Ref 11:275). These studies of visual effects have been based on a reference frame which has a constant velocity and is not accelerating.

Additional studies have developed the dynamic equations of motion for vehicles in relativistic flight. Most of the studies were developed from the viewpoint of an observer located on an inertial reference frame with the spacecraft traveling at constant velocity. However, there have been a few which considered the viewpoint of an observer located on an accelerating reference frame which required the use of the theory of general relativity (Refs 4,5). The viewpoint of an observer on an accelerating reference frame is equivalent to the viewpoint of the pilot of a relativistic spacecraft which is being accelerated.

Problem Statement

The basic problem is that no general navigational methods have been developed for spacecraft traveling at relativistic speeds. The navigation problem must be solved before flight to the stars can be considered a realistic goal. Separate parts of the navigation problem have been studied independently and with varying assumptions, but there has been no application or combination of these separate studies to develop a method of navigation.

Navigation has been defined as the process of directing the movement of a vehicle from one point to another (Ref 10:1). This process consists of two operations: determining the present position and velocity of the vehicle relative to a known reference system and modifying the course of the vehicle to reach a destination. The primary purpose of this study is to develop general navigational techniques which can be used to determine the position and velocity of a spacecraft traveling at relativistic speeds. The goal is to develop closed form expressions for the inertial position and velocity of the spacecraft in terms of parameters which can be measured by an observer located on-board the spacecraft.

Assumptions

The assumptions made in this analysis are:

1. All necessary measurements and information processing will be performed on the spacecraft.

2. The guidance system is accurate enough to maintain a one-dimensional direction of travel along the X axis toward the destination star.

3. There are no gravitational or drag forces acting on the spacecraft.

4. The position of the sun and the earth are coincident and located at the origin of the inertial coordinate system.

5. The stars used for measurements to determine the spacecraft position and velocity are located in the XY plane.

These assumptions were made in order to reduce the complexity of the problem so that a reasonable model could be developed for this conceptual study.

Method of Approach

The basic approach is to model the flight path of the spacecraft using a Cartesian coordinate system centered at the sun and with the X axis directed toward the destination star as shown in Figure 1. Since the guidance system is assumed to be accurate enough to maintain the Y and Z components of position and velocity at zero, the problem is reduced to a one-dimensional analysis. Two cases will be investigated: one where the spacecraft is traveling at constant velocity and the second where the spacecraft is undergoing a constant acceleration as measured by an accelerometer located on the spacecraft.

The transformation equations (Ref 7) from the theory of special and general relativity are used to develop a wavelength shift equation, which relates the inertial wavelength λ of light emitted from a star to the wavelength λ' measured by an observer on the spacecraft, and a relativistic aberration equation which relates the inertial angle θ of a star to the position angle θ' as measured by an observer on the spacecraft for each case. The wavelength shift equation is used as

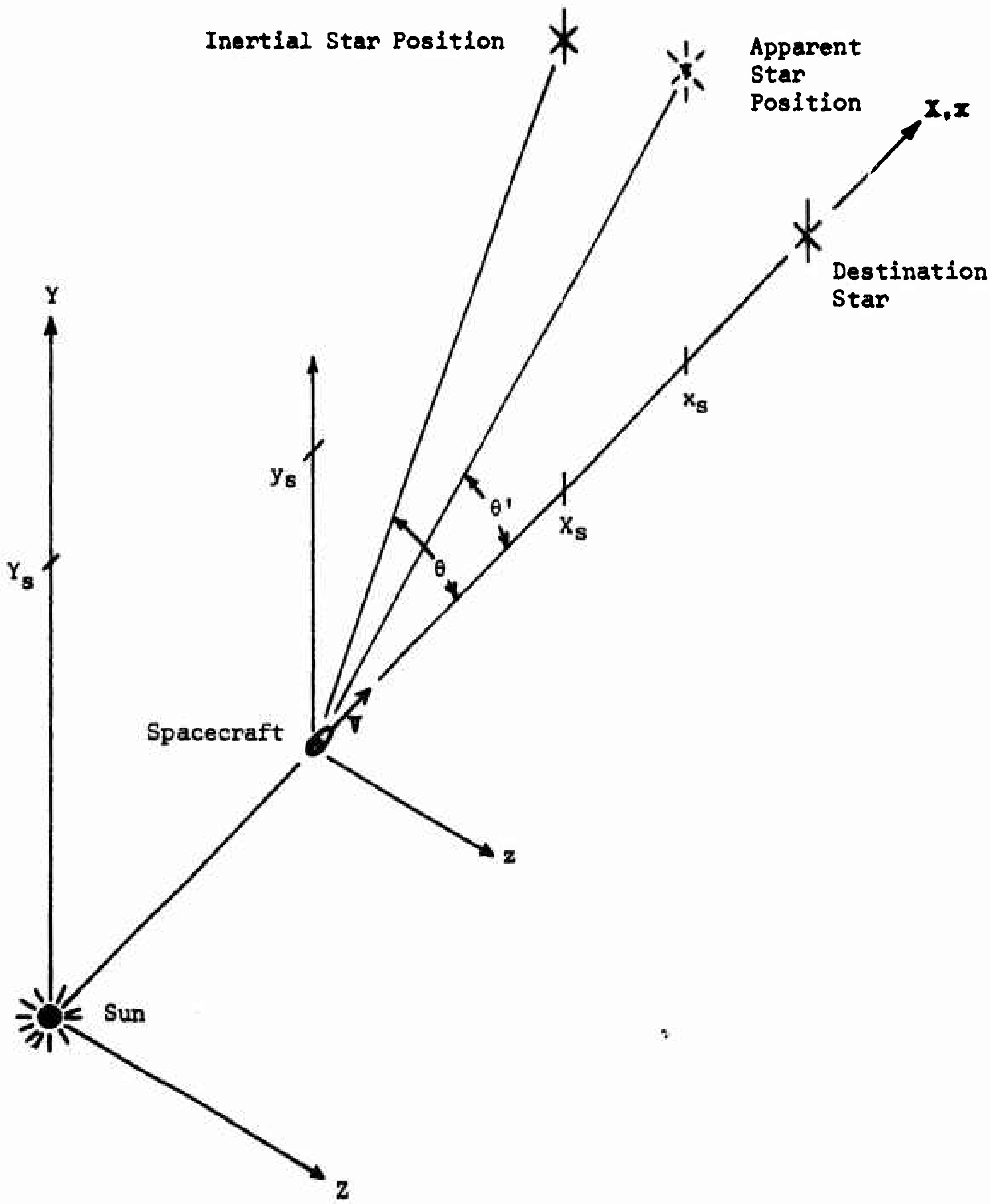

Figure 1. Coordinate Systems and Geometry for Relativistic Flight

the basis for developing a velocity measuring technique and the relativistic aberration equation is used as the basis for developing a position measuring technique. An error analysis is made to determine how the measurements should be taken so as to minimize the error in determining the inertial position and velocity of the spacecraft.

II. Constant Velocity Flight Equations

The purpose of this chapter is to develop and summarize the equations for constant velocity flight which will be needed in later chapters. The equations are developed using the theory of special relativity. A concise summary of the theory of relativity as applied to spacecraft flight is contained in Chapter 2 of Reference 1. A complete and general development of the theory of special relativity may be found in Reference 7.

Lorentz Transformations

The Lorentz transformations relate the coordinates (X,Y,Z,T) of an event, as measured by an observer at the origin of an inertial coordinate system, and the coordinates (x,y,z,t) of the same event as measured by a second observer located at the origin of a coordinate system which is moving with a constant velocity V, relative to the inertial system, in the X direction. The derivation of the Lorentz transformation is based on the fact that the speed of light is the same for both observers (Ref 7:36-40). In this study the inertial observer is located at the sun, or earth since both positions are assumed coincident, and the second observer is located on the spacecraft.

The Lorentz transformation, assuming the system origins are coincident at T = t = 0, is (Ref 7:40)

$$x = \frac{X - VT}{[1-(V/c)^2]^{1/2}} \tag{2-1}$$

$$y = Y \tag{2-2}$$

$$z = Z \tag{2-3}$$

$$t = \frac{T - VX/c^2}{[1-(V/c)^2]^{1/2}} \tag{2-4}$$

The inverse transformation is (Ref 7:40)

$$X = \frac{x + Vt}{[1-(V/c)^2]^{1/2}} \tag{2-5}$$

$$Y = y \tag{2-6}$$

$$Z = z \tag{2-7}$$

$$T = \frac{t + Vx/c^2}{[1-(V/c)^2]^{1/2}} \tag{2-8}$$

An observer on the earth describes the inertial position of the spacecraft and spacecraft time by

$$X_v = VT \tag{2-9}$$

$$t_v = T[1-(V/c)^2]^{1/2} \tag{2-10}$$

These equations are obtained from equations (2-1) and (2-4) by setting $x = 0$ in equation (2-1) and then substituting equation (2-9) into equation (2-4) to obtain equation (2-10).

An observer on the spacecraft describes the position of the earth and earth time by

$$x_e = -Vt \tag{2-11}$$

$$T_e = t[1-(V/c)^2]^{1/2} \tag{2-12}$$

These equations are obtained from equations (2-5) and (2-8) by setting $X = 0$ in equation (2-5) and then substituting equation (2-11) into equation (2-8) to obtain equation (2-12).

Relativistic Aberration

The observer on the spacecraft will observe several visual effects which result from flight at relativistic speeds (Refs 8,9,11). One of these effects is the apparent shift in the position of stars. An observer specifies the position of a star by the location at which he sees the light emitted by a star. An observer on the earth specifies the location of a star by measuring its inertial position angle θ while an observer on the spacecraft locates the same star by measuring its apparent position angle θ'. The relativistic aberration equation relates the inertial position angle θ to the apparent position angle θ' and is derived from the inverse Lorentz transformation equations. The necessary geometry is shown in Figure 1.

Assume that a photon of light is emitted from a star. The location of this photon of light is specified as (X,Y,T) by the earth observer and (x,y,t) by the observer on the spacecraft. These locations are related by equations (2-5), (2-6), and (2-8). Differentiation of equations (2-5), (2-6), and (2-8) yields

$$[1-(V/c)^2]^{1/2}\, dX = dx + V dt \tag{2-13}$$

$$[1-(V/c)^2]^{1/2}\, dT = dt + (V/c^2)\, dx \tag{2-14}$$

$$dY = dy \tag{2-15}$$

Forming dX/dT from equations (2-13) and (2-14) and dY/dT from equations (2-15) and (2-14) results in

$$\frac{dX}{dT} = \frac{dx/dt + V}{1 + (V/c^2)\ dx/dt} \tag{2-16}$$

$$\frac{dY}{dT} = \frac{[1-(V/c)^2]^{1/2}\ dy/dt}{1 + (V/c^2)\ dx/dt} \tag{2-17}$$

The velocity of light c is the same in both coordinate systems and since the light is traveling from the star toward the origins of the coordinate systems, then for light from a star

$$dx/dt = -\ c \cos \theta' \tag{2-18}$$

$$dy/dt = -\ c \sin \theta' \tag{2-19}$$

$$dX/dT = -\ c \cos \theta \tag{2-20}$$

$$dY/dT = -\ c \sin \theta \tag{2-21}$$

Substituting equations (2-18) through (2-21) into equations (2-16) and (2-17) yields

$$-\ c \cos \theta = \frac{-\ c \cos \theta' + V}{1 - (V/c) \cos \theta'} \tag{2-22}$$

$$-\ c \sin \theta = \frac{[1-(V/c)^2]^{1/2}\ (-c \sin \theta')}{1 - (V/c) \cos \theta'} \tag{2-23}$$

Dividing equation (2-23) by equation (2-22) yields

$$\tan \theta = \frac{[1-(V/c)^2]^{1/2} \sin \theta'}{\cos \theta' - V/c} \qquad (2\text{-}24)$$

This is the relativistic aberration equation for constant velocity flight which relates the inertial angle θ of the position of a star to the angle θ' measured by an observer on the spacecraft.

Wavelength Shift

In addition to a shift in star positions, another visual effect the observer on the spacecraft will experience is a shift in the wavelength of the light emitted by stars. For higher velocities certain stars will seem to disappear as their apparent wavelength shifts outside the visual spectrum. The colors of the visible stars will range through the entire visual spectrum from violet to red (Refs 8,9,11).

The wavelength shift equation is (Ref 11:274)

$$\frac{\lambda'}{\lambda} = \frac{1 - (V/c) \cos \theta'}{[1-(V/c)^2]^{1/2}} \qquad (2\text{-}25)$$

which relates the inertial wavelength λ of light emitted from a star to the wavelength λ' measured by an observer on the spacecraft. For example, the wavelength shift of light emitted from the sun as seen by an observer on the spacecraft is given by (Ref 11:274)

$$\frac{\lambda'}{\lambda} = \left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \qquad (2\text{-}26)$$

Equation (2-26) is obtained from equation (2-25) by observing that $\cos \theta' = -1$ for the sun. As the spacecraft velocity increases the

apparent wavelength λ' shifts toward the red end of the visual spectrum and then into the infared region of the electromagnetic spectrum.

To the observer on the spacecraft the wavelength shift of the light emitted from the destination star will be (Ref 11:274)

$$\frac{\lambda'}{\lambda} = \left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \tag{2-27}$$

Equation (2-27) is obtained from equation (2-25) by observing that $\cos \theta' = 1$ for the destination star. As the spacecraft velocity increases, the apparent wavelength of light emitted from the destination star shifts toward the blue end of the visual spectrum and then into the ultraviolet region of the electromagnetic spectrum.

III. Constant Acceleration Flight Equations

The purpose of this chapter is to develop and summarize the equations for constant acceleration flight which will be needed in later chapters. These equations are developed using the theory of general relativity. A concise summary of the theory of general relativity as applied to spacecraft flight is contained in Chapter 3 of Reference 1. A complete and general development of the theory of general relativity may be found in Reference 7.

Transformation Equations

The transformation equations in this case are expressions which relate the coordinates (X,Y,Z,T) of an event as seen by an observer located at the origin of the inertial coordinate system and the coordinates (x,y,z,t) of the same event as seen by a second observer located at the origin of non-inertial coordinate system which is moving with a constant acceleration g, as measured by an accelerometer fixed in the moving system, in the X direction. In this study the inertial observer is located at the sun, or earth since both positions are assumed to be coincident, and the second observer is located on the spacecraft. The general transformation equations, assuming the origins are coincident and have zero relative velocity at $T = t = 0$, are (Ref 7:256)

$$X = (c^2/g)[\cosh(gt/c)-1] + x \cosh(gt/c) \quad (3\text{-}1)$$

$$Y = y \quad (3\text{-}2)$$

$$Z = z \quad (3\text{-}3)$$

$$T = (c/g + x/c) \sinh (gt/c) \qquad (3\text{-}4)$$

An observer located on earth describes the inertial position of the spacecraft and spacecraft time by

$$X_v = (c^2/g) [\cosh (gt/c)-1] \qquad (3\text{-}5)$$

$$t_v = (c/g) \sinh^{-1} (gT/c) \qquad (3\text{-}6)$$

These equations are obtained from equations (3-1) and (3-4) by setting $x = 0$ in both equations. The observer on earth describes the spacecraft velocity as

$$V = dX/dT = c \tanh (gt/c) \qquad (3\text{-}7)$$

An observer located on the spacecraft describes the position of the earth by

$$x_e = (c^2/g) [\operatorname{sech} (gt/c)-1] \qquad (3\text{-}8)$$

Equation (3-8) is obtained from equation (3-1) by setting $X = 0$. As the time t approaches very large values, the position of the earth x_e approaches a limiting value of $-c^2/g$ which is the location of the "singular wall" (Ref 7:258). At the singular wall the velocity of light apparently approaches zero and therefore no signals or any information relayed by electromagnetic waves will ever reach the spacecraft when the information is transmitted from locations behind the singular wall. The singular wall exists only for the observer located on the rocket, however an observer on earth also notes that

after a certain time information which he sends does not appear to reach the spacecraft (Ref 4).

The observer on the spacecraft describes time on the earth, as a function of spacecraft time t, by

$$T_e = (c/g) \tanh (gt/c) \tag{3-9}$$

Equation (3-9) is obtained by substituting equation (3-8) into equation (3-4) and simplifying the result.

The observer on the spacecraft will also experience various visual effects similar to those described in chapter II for constant velocity flight. However, in constant acceleration flight the observer would not only observe apparent shifts in stellar position and wavelength, but he would also observe that the speed of light is no longer constant. In fact he would observe that the apparent speed of light w can be greater than the inertial speed of light, $c = 3\times10^8$ meters/second.

Apparent Speed of Light

The basic invariant in the theory of special relativity is the speed of light c. For general relativity a new invariant quantity ds^2 is defined (Ref 7:99). In the case of constant acceleration flight the invariant, in the xy plane, is given by (Ref 7:255)

$$ds^2 = dx^2 + dy^2 - c^2(1 + gx/c^2)^2\, dt^2 \tag{3-10}$$

For light waves $ds^2 = 0$ and equation (3-10) becomes, after slight rearrangement

$$(dx/dt)^2 + (dy/dt)^2 = c^2(1 + gx/c^2)^2 \tag{3-11}$$

which implies that

$$w^2 = c^2(1 + gx/c^2)^2 \tag{3-12}$$

From equation (3-12) the speed of light w in the spacecraft centered system is a function of the position x and is not a constant.

Relativistic Aberration

The relativistic aberration equation for constant acceleration flight is developed using an approach similar to that used in chapter II for constant velocity flight. Again the geometry is as shown in Figure 1.

Differentiation of equation (3-1), (3-2), and (3-4) yields

$$dX = (c + gx/c)\sinh(gt/c)\,dt + \cosh(gt/c)\,dx \tag{3-13}$$

$$dY = dy \tag{3-14}$$

$$dT = (1/c)\sinh(gt/c)\,dx + (1 + gx/c^2)\cosh(gt/c)\,dt \tag{3-15}$$

Forming dX/dT from equations (3-13) and (3-15) yields

$$\frac{dX}{dT} = \frac{(c + gx/c)\sinh(gt/c) + \cosh(gt/c)\,dx/dt}{(1/c)\sinh(gt/c)\,dx/dt + (1 + gx/c^2)\cosh(gt/c)} \tag{3-16}$$

and dY/dT from equations (3-14) and (3-15) results in

$$\frac{dY}{dT} = \frac{dy/dt}{(1/c)\sinh(gt/c)\,dx/dt + (1 + gx/c^2)\cosh(gt/c)} \tag{3-17}$$

The speed of light in the inertial system is c and the light is traveling from the star toward the origin so that

$$dX/dT = -c\cos\theta \tag{3-18}$$

$$dY/dT = -c\sin\theta \tag{3-19}$$

Now in the spacecraft coordinate system the speed of light w varies as shown in the preceding section. Again the light is traveling from the star toward the origin of the system and

$$\frac{dx}{dt} = -w\cos\theta' = -(c + gx/c)\cos\theta' \tag{3-20}$$

$$\frac{dy}{dt} = -w\sin\theta' = -(c + gx/c)\sin\theta' \tag{3-21}$$

Substituting equations (3-18), (3-19), (3-20), and (3-21) into equations (3-16) and (3-17) yields

$$-c\cos\theta = \frac{(c + gx/c)\sinh(gt/c) - \cosh(gt/c)(c + gx/c)\cos\theta'}{-(1/c)\sinh(gt/c)(c + gx/c)\cos\theta' + (1 + gx/c^2)\cosh(gt/c)} \tag{3-22}$$

$$-c\sin\theta = \frac{-(c + gx/c)\sin\theta'}{-(1/c)\sinh(gt/c)(c + gx/c)\cos\theta' + (1 + gx/c^2)\cosh(gt/c)} \tag{3-23}$$

Dividing equation (3-23) by equation (3-22) results in

$$\tan\theta = \frac{\sin\theta'}{\cosh(gt/c)\cos\theta' - \sinh(gt/c)} \tag{3-24}$$

or

$$\tan\theta = \frac{\sin\theta' \ \mathrm{sech}(gt/c)}{\cos\theta' - \tanh(gt/c)} \tag{3-25}$$

Now observing that

$$\mathrm{sech}(gt/c) = [1 - \tanh^2(gt/c)]^{1/2} \tag{3-26}$$

$$\tanh(gt/c) = V/c \tag{3-27}$$

and substituting equations (3-26) and (3-27) into equation (3-25) yields

$$\tan\theta = \frac{\sin\theta' \ [1-(V/c)^2]^{1/2}}{\cos\theta' - V/c} \tag{3-28}$$

which is the relativistic aberration equation for constant acceleration flight. Equation (3-28) is identical to the relativistic aberration equation for constant velocity flight, equation (2-24). However, in constant acceleration flight the velocity V and apparent position angle θ' are continuously changing with time due to the acceleration of the spacecraft and therefore the inertial position angle θ computed from equation (3-28) is an instantaneous value only.

Wavelength Shift

The wavelength shift equation in constant acceleration flight is derived using the general principle of relativity that all systems of reference are equivalent with respect to the formulation of the fundamental laws of physics (Ref 7:220). By this principle the phase of a plane wave is an invariant and can be given by

$$P = f\left(t - \frac{\xi \cos \gamma + \eta \sin \gamma}{b}\right) \tag{3-29}$$

where

f = the wave frequency

b = the wave speed

γ = the angle the wave normal makes with the ξ axis

ξ,η = the location of the wave

Equating the phase of a light wave in the inertial coordinate system to the phase of the same light wave in the spacecraft coordinate system results in

$$f\left(T - \frac{X \cos \alpha + Y \sin \alpha}{c}\right) = f'\left(t - \frac{x \cos \alpha' + y \sin \alpha'}{w}\right) \tag{3-30}$$

Substituting for X, Y, and T from equations (3-1), (3-2), and (3-4) and for w from equation (3-12) yields

$$f\left\{\left(\frac{c}{g} + \frac{x}{c}\right)\sinh\left(\frac{gt}{c}\right) - \frac{c}{g}\left[\cosh\left(\frac{gt}{c}\right)-1\right] \cos\alpha - \frac{x}{c} \cosh\left(\frac{gt}{c}\right)\cos\alpha - \frac{y}{c} \sin\alpha\right\}$$

$$= f'\left[t - \frac{x \cos \alpha'}{c(1 + gx/c^2)} - \frac{y \sin \alpha'}{c(1 + gx/c^2)}\right] \tag{3-31}$$

Equating the coefficients of y on both sides of equation (3-31) yields

$$-(f/c) \sin \alpha = -(f'/c) \frac{\sin \alpha'}{(1 + gx/c^2)} \tag{3-32}$$

or rearranging

$$f/f' = \frac{\sin \alpha'}{\sin \alpha \, (1 + gx/c^2)} \tag{3-33}$$

Noting that $\alpha' = \theta' + \pi$ and $\alpha = \theta + \pi$ and using equation (3-23) results in

$$\frac{\sin \alpha'}{\sin \alpha} = \frac{\sin \theta'}{\sin \theta} = \cosh\left(\frac{gt}{c}\right) - \sinh\left(\frac{gt}{c}\right) \cos \theta' \tag{3-34}$$

so that

$$f/f' = \frac{\cosh(gt/c) - \sinh(gt/c) \cos \theta'}{1 + gx/c^2} \tag{3-35}$$

However, a relationship for the wavelength shift λ'/λ is desired instead of a frequency shift relation and therefore a transformation is necessary. For the inertial system

$$f\lambda = c \tag{3-36}$$

while in the spacecraft centered system

$$f'\lambda' = w \tag{3-37}$$

The measurements of frequency f' and wavelength λ' will be made at the origin of the spacecraft centered system and from equation (3-12) $w = c$ when $x = 0$ so that

$$f\lambda = f'\lambda' \tag{3-38}$$

Thus equation (3-35) becomes

$$\frac{\lambda'}{\lambda} = \frac{1 - \tanh(gt/c)\cos\theta'}{\mathrm{sech}(gt/c)} \tag{3-39}$$

Now substituting equations (3-26) and (3-27) into equation (3-39) results in

$$\frac{\lambda'}{\lambda} = \frac{1 - (V/c)\cos\theta'}{[1-(V/c)^2]^{1/2}} \tag{3-40}$$

Equation (3-40) is identical to equation (2-25) the wavelength shift equation for constant velocity flight. Equating the coefficients of x in equation (3-31) and rearranging with similar substitutions also results in equation (3-40).

For the sun $\cos\theta' = -1$ and equation (3-40) becomes

$$\frac{\lambda'}{\lambda} = \frac{1 + V/c}{[1-(V/c)^2]^{1/2}} = \left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \tag{3-41}$$

Equation (3-41) is identical to equation (2-26) and as the spacecraft velocity increases the apparent wavelength λ' again shifts from the visual spectrum into the infared region of the electromagnetic spectrum.

For the destination star $\cos\theta' = 1$ so that equation (3-40) becomes

$$\frac{\lambda'}{\lambda} = \frac{1 - V/c}{[1-(V/c)^2]^{1/2}} = \left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \tag{3-42}$$

Equation (3-42) is identical to equation (2-27) and as the spacecraft velocity increases the apparent wavelength of the destination star λ' again shifts from the visual spectrum into the ultraviolet region of the electromagnetic spectrum. When the velocity is zero the apparent wavelength is equal to the inertial wavelength in equations (3-40), (3-41), and (3-42).

IV. Development of Methods for Determining Navigational Parameters

In this chapter closed form expressions are developed for measuring the inertial position and velocity of the spacecraft by measuring certain parameters with instruments located on the spacecraft. These expressions are developed in two sections: the first section is for spacecraft flight at constant velocity and the second for spacecraft flight at constant acceleration as measured by an accelerometer on the spacecraft. Also in this chapter it is shown that the expressions developed do not restrict the entire spacecraft flight to either constant velocity or constant acceleration. In addition, two methods for determining the direction of travel of the spacecraft are introduced.

Constant Velocity Methods

In planning a flight it is necessary that a method of determining the position as a function of time be available. In this analysis it is assumed that a velocity has been specified and time will be measured on the spacecraft. From equations (2-9) and (2-10) it can be easily shown that

$$X_V = Vt/[1-(V/c)^2]^{1/2} \tag{4-1}$$

Thus, the position of the spacecraft at any time t can be computed from equation (4-1) when the velocity is specified.

During the actual flight it will be necessary to measure the spacecraft velocity. One method of measuring the velocity is to measure the apparent wavelength λ' and apparent position angle θ' of

a star, assuming the inertial wavelength λ of light emitted from the star is known. Beginning with the wavelength shift equation

$$\frac{\lambda'}{\lambda} = \frac{1 - (V/c)\cos\theta'}{[1-(V/c)^2]^{1/2}} \qquad (2\text{-}25)$$

Squaring and rearranging results in

$$(\lambda'/\lambda)^2[1-(V/c)^2] = 1 - 2(V/c)\cos\theta' + (V/c)^2\cos^2\theta \qquad (4\text{-}2)$$

or

$$[\cos^2\theta' + (\lambda'/\lambda)^2](V/c)^2 - 2\cos\theta'(V/c) + [1-(\lambda'/\lambda)^2] = 0 \qquad (4\text{-}3)$$

Now solving for V/c yields

$$\frac{V}{c} = \frac{\cos\theta'}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \pm \left\{\frac{\cos^2\theta' - \left[1 - \left(\frac{\lambda'}{\lambda}\right)^2\right]\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2}\right\}^{1/2} \qquad (4\text{-}4)$$

Rearranging equation (4-4) results in

$$\frac{V}{c} = \frac{\cos\theta' \pm \left\{\cos^2\theta' - \cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\left[\left(\frac{\lambda'}{\lambda}\right)^2 + \cos^2\theta' - 1\right]\right\}^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \qquad (4\text{-}5)$$

Further simplification yields

$$\frac{V}{c} = \frac{\cos\theta' \pm \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \tag{4-6}$$

In order for this expression to be valid, the roots must always be real which implies that

$$(\lambda'/\lambda)^2 \geq \sin^2\theta' \tag{4-7}$$

must always be true. To verify this condition equation (2-25) is rearranged as

$$\left(\frac{\lambda'}{\lambda}\right)^2 = \frac{[1-(V/c)\cos\theta']}{1-(V/c)^2} = \frac{1 - 2(V/c)\cos\theta' + (V/c)^2\cos^2\theta'}{1-(V/c)^2} \tag{4-8}$$

Substituting equation (4-8) into equation (4-7) yields

$$1 - 2(V/c)\cos\theta' + (V/c)^2\cos^2\theta' \geq \sin^2\theta' - (V/c)^2\sin^2\theta' \tag{4-9}$$

or

$$1 - \sin^2\theta' - 2(V/c)\cos\theta' + (V/c)^2(\sin^2\theta' + \cos^2\theta') \geq 0 \tag{4-10}$$

or

$$\cos^2\theta' - 2(V/c)\cos\theta' + (V/c)^2 \geq 0 \tag{4-11}$$

Factoring equation (4-11) results in

$$(\cos\theta' - V/c)^2 \geq 0 \tag{4-12}$$

Since Equation (4-12) is always true, the roots of equation (4-6) are always real.

The switching point of the sign in equation (4-6) occurs when

$$(\lambda'/\lambda) = \sin^2\theta' \tag{4-13}$$

Using the equality from equation (4-12) results in the equivalent condition that

$$(\cos\theta' - V/c)^2 = 0 \tag{4-14}$$

Therefore the switching point occurs at

$$\cos\theta' = V/c \tag{4-15}$$

For the sun, from equation (2-26)

$$(\lambda'/\lambda)^2 = \frac{1 + V/c}{1 - V/c} \tag{4-16}$$

Rearranging yields

$$V/c = \frac{(\lambda'/\lambda)^2 - 1}{1 + (\lambda'/\lambda)^2} \tag{4-17}$$

To obtain this result from equation (4-6) the plus sign must be used. Therefore, switching occurs at $\cos\theta' = V/c$ and the plus sign is used if $\cos\theta' < V/c$ and the negative sign is used if $\cos\theta' > V/c$ is true.

In order to determine the position of the spacecraft it is necessary to determine the inertial angle θ between the known position of a star (X_s, Y_s) and the X axis as shown in Figure 2.

From Figure 2

$$X_v = X_s - Y_s/\tan\theta \tag{4-18}$$

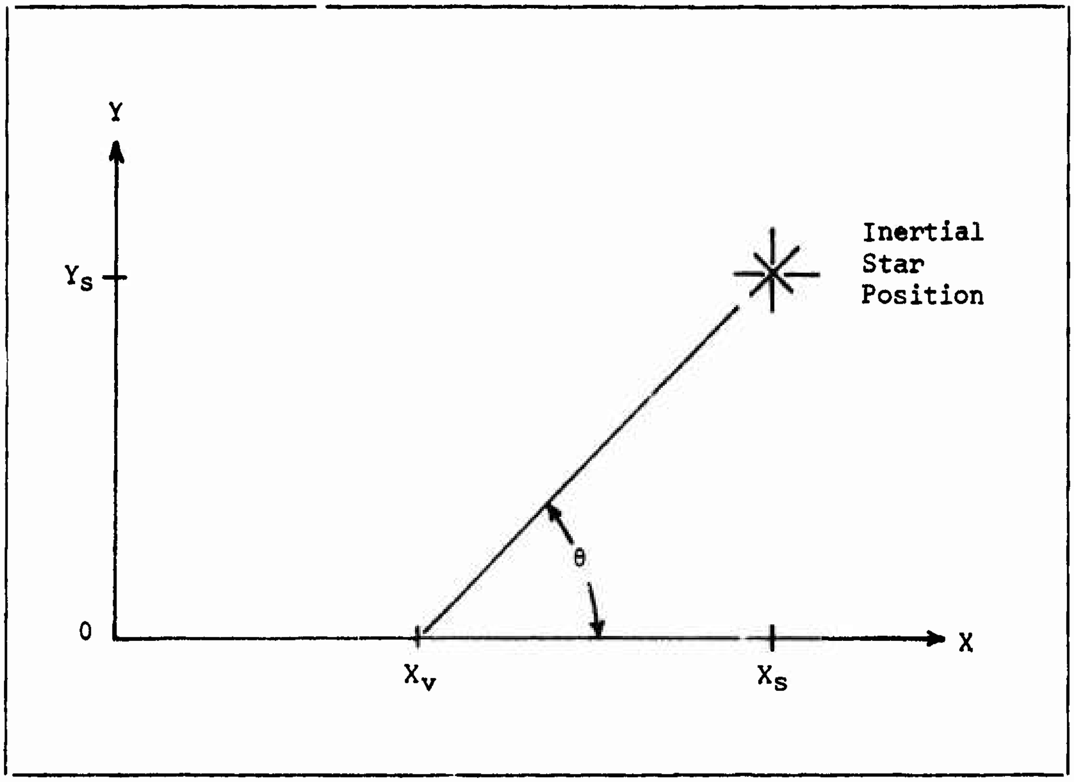

Figure 2. Inertial Position Geometry

Substituting for tan θ from the relativistic aberration equation (2-24) results in

$$X_v = X_s - \frac{Y_s\ (\cos\theta' - V/c)}{\sin\theta'\ [1-(V/c)^2]^{1/2}} \tag{4-19}$$

Therefore, by measuring the apparent angle θ' to a star and the velocity of the spacecraft using equation (4-6), it is possible to determine the inertial position of the spacecraft, assuming the inertial position (X_s, Y_s) of the star is known.

Constant Acceleration Methods

In constant acceleration flight it is also necessary for flight planning to have a method of determining position and velocity as a

function of the acceleration and time as measured on the spacecraft. For flight planning it is assumed that the acceleration is specified and time is measured on the spacecraft. The inertial position is computed from equation (3-5) which is

$$X_v = (c^2/g)\,[\cosh(gt/c)-1] \tag{3-5}$$

The velocity is computed from equation (3-7) which is

$$V = c\,\tanh(gt/c) \tag{3-7}$$

During the actual flight, as in the constant velocity case, the spacecraft velocity is measured by measuring the apparent wavelength λ' and apparent position angle θ' of a star. Since the wavelength shift equation for constant acceleration flight is identical to the wavelength shift equation in constant velocity flight, the development of the velocity measuring equation for constant acceleration flight is identical to the development of equation (4-6) in the preceding section. Therefore, the velocity is determined using

$$\frac{V}{c} = \frac{\cos\theta' \pm \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \tag{4-6}$$

The plus sign is used if $\cos\theta' < V/c$ and the negative sign if $\cos\theta' > V/c$.

Since the relativistic aberration equation is also the same for both constant velocity and constant acceleration flight, the position of a spacecraft in constant acceleration flight is determined from equation (4-19) which was developed for constant velocity flight but which is also true for constant acceleration flight.

Modification of Flight Equations

Although the transformation equations for both constant velocity and constant acceleration flight were based on the assumption that the origins of the inertial and spacecraft coordinate systems were coincident at T = t = 0 this does not restrict the entire spacecraft flight to either constant velocity or constant acceleration flight. For example, a typical interstellar mission may include an acceleration phase during which a certain velocity is reached, then a coasting phase at a constant velocity, and then a deceleration phase which terminates with the spacecraft reaching zero velocity at the destination star. This example flight profile is shown in Figure 3.

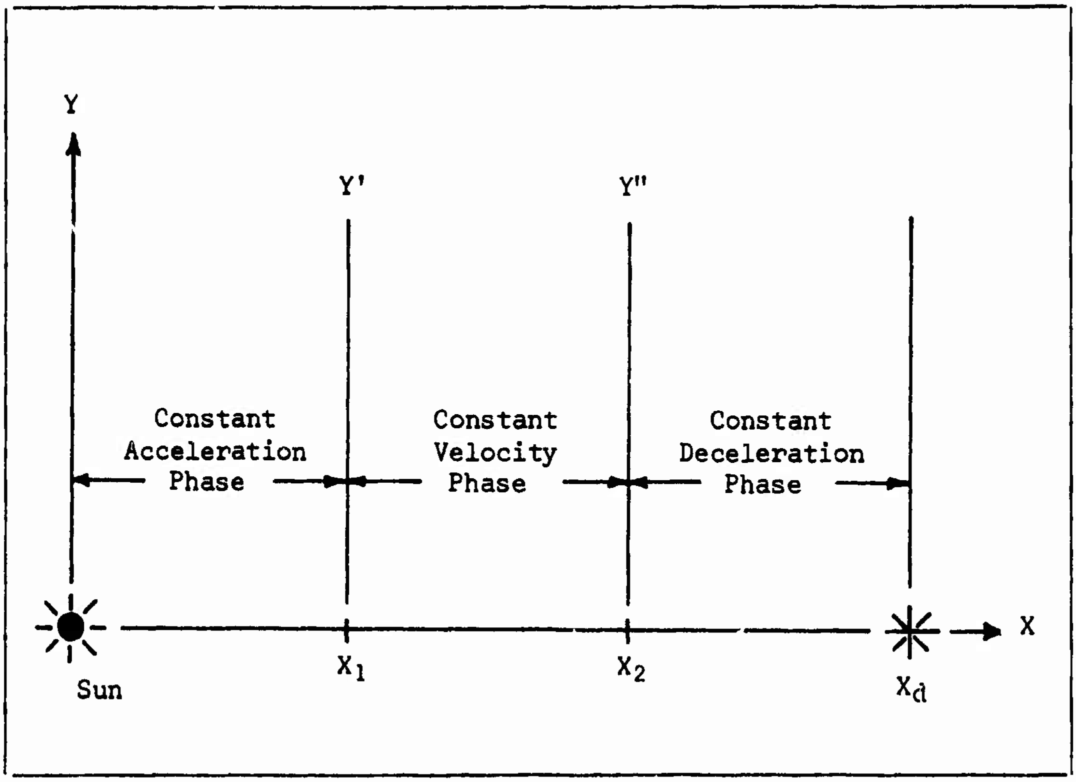

Figure 3. Example Flight Profile

In order to modify the flight equations which are dependent on position X and time t to allow for a flight profile of this type a coordinate transformation is necessary. Note that no modification of previous equations for constant acceleration flight is required for spacecraft flight between $X = 0$ and $X = X_1$. The coordinate transformation is accomplished by introducing two new inertial coordinate systems (X', Y' centered at X_1 and X'', Y'' centered at X_2) and by measuring time from the time of arrival of the spacecraft at X_1 or X_2 as appropriate. Therefore, when the spacecraft is located between X_1 and X_2 the transformation equations are

$$X = X_1 + X' \tag{4-20}$$

$$Y = Y' \tag{4-21}$$

$$t = t_1 + t' \tag{4-22}$$

When the spacecraft is between X_2 and X_d the transformation equations are

$$X = X_2 + X'' \tag{4-23}$$

$$Y = Y'' \tag{4-24}$$

$$t = t_2 + t'' \tag{4-25}$$

For the constant velocity phase of the flight, with the spacecraft between X_1 and X_2 in this case, some modification of the previously derived equations may be necessary based on the above transformation equations. In order to compute the spacecraft position equation (4-1) must be modified using equations (4-20) and (4-22) to

$$X_v = X_1 + \frac{V(t-t_1)}{[1-(V/c)^2]^{1/2}} \tag{4-26}$$

However, the expression for measuring the velocity, equation (4-6), is not an explicit function of either position X or time t and therefore it does not have to be modified.

No modification of the position measuring expression, equation (4-19), is necessary although this is not readily apparent. Substituting the transformation equations (4-20) and (4-21) into equation (4-19) results in

$$X'_v + X_1 = X'_s + X_1 - \frac{Y'_s\ (\cos\theta' - V/c)}{\sin\theta'\ [1-(V/c)^2]^{1/2}} \tag{4-27}$$

This equation is independent of X_1, as the X_1 terms cancel, and since $Y_s = Y'_s$ the equation can be used to determine the spacecraft inertial position X_v. Therefore, no modification of the expressions for measuring the spacecraft inertial position and velocity is necessary and those techniques developed previously can be used without modification during any period of the flight when the spacecraft is traveling at a constant velocity.

For the constant deceleration phase of the flight, with the spacecraft between X_2 and X_d, some modification of the previously developed expressions for constant acceleration flight may be necessary. In order to compute the spacecraft position equation (3-5) must be modified using equations (4-23) and (4-25) which results in

$$X_v = X_2 + \frac{c^2}{|g|}\left\{\cosh\left[\frac{g(t-t_2)}{c}\right] - 1\right\} \tag{4-28}$$

The absolute value of the acceleration g must be used during deceleration as position must increase during both acceleration and deceleration phases of the flight, that is the vehicle does not begin to travel backwards during deceleration pericds. In order to compute the velocity during the deceleration phase equation (3-7) must be modified to read

$$V = V_2 + c \tanh\left[\frac{g(t-t_2)}{c}\right] \tag{4-29}$$

Since the velocity and position measuring methods for constant acceleration flight are identical to the methods developed for constant velocity flight, no modification is necessary and the velocity and inertial position of the spacecraft can be measured during any constant acceleration, or deceleration, phase of the flight using equation (4-6) or (4-19) as appropriate.

Direction Determination Methods

In addition to determining the magnitude of the spacecraft velocity, or speed, the direction of travel must also be determined. Two different techniques are presented: one which depends on tracking the position of the destination star and a second which depends on determining the location of stars which do not appear to shift in wavelength to an observer on the spacecraft. Both these methods would

depend upon automated tracking instruments and computers being available on the spacecraft.

The first method is based on tracking the inertial position of the destination star and directing the thrust of the spacecraft, during thrusting periods, toward the destination star. The tracking is complicated by the continual shift in wavelength, most likely out of the visual spectrum, of light from the destination star. However, the position may be tracked by precomputing the expected wavelength shift and designing tracking instruments to follow a pre-programmed search for the proper wavelength. If the actual flight deviates from the specified flight plan, corrections could be made to provide new wavelength information to the tracking instrument based on the updated flight information. This method could be used most effectively during the constant acceleration phase of a flight.

A second direction finding technique is also based on the wavelength shift of light emitted from the stars. For each velocity V there is a value of the apparent position angle θ' for which there is no shift in wavelength so that $\lambda' = \lambda$. This value of θ' is derived from equation (2-25) as follows

$$\lambda'/\lambda = \frac{1-(V/c)\cos\,\theta'}{[1-(V/c)^2]^{1/2}} = 1 \tag{2-25}$$

Solving for cos θ' yields

$$\cos\,\theta' = \frac{1 - [1-(V/c)^2]^{1/2}}{V/c} \tag{4-30}$$

Therefore there will appear to be a circle of stars ahead of the spacecraft which apparently do not shift in wavelength. By sweeping the entire forward hemisphere on a periodic basis and using a computer to store the wavelengths measured at each position for each sweep, it would be possible to determine and plot on a display screen the circle of stars, and computed center of this circle, for which the wavelengths did not change from the previous sweep. By keeping the center of this circle and the position of the destination star coincident the spacecraft would always be directed toward the destination star. This information can also be used to determine the spacecraft velocity by determining the angle θ' and then computing the velocity using

$$V/c = \frac{2 \cos \theta'}{1 + \cos^2\theta'} \tag{4-31}$$

which can be obtained from equation (4-30).

The direction of travel can also be determined by measuring the inertial spacecraft position at discrete time intervals and plotting these positions. If necessary it would be possible to use a predictor technique within the computer and possibly make course corrections based on the predicted positions.

V. Analysis of Methods for Determining Navigational Parameters

In this chapter the expressions and methods developed in chapter IV are analyzed to determine the best methods of measuring the inertial position and velocity of the spacecraft. An error analysis of each expression is made to determine the relative error caused by an error in the value of each of the measured parameters. The error expressions are developed using a Taylor's Series expansion for the error term assuming that the individual errors in measurement are small enough that the second and higher order terms may be neglected. Only the final error expressions are presented in this chapter. The complete differentiation and development of these expressions are contained in Appendix A. When numerical values are needed to develop the plots in this chapter it is assumed that

$$g = 1 \text{ light year/year}^2 \simeq 1 \text{ "g"}$$

$$c = 1 \text{ light year/year}$$

Distances are measured in light years and time is measured in years.

Constant Velocity Error Analysis

For spacecraft flight at constant velocity the inertial position is computed from

$$X_v = \frac{Vt}{[1-(V/c)^2]^{1/2}} \tag{4-1}$$

where the velocity V and the time t are the measured parameters.

Assume there is an error δt in measuring the time. The resulting error in position is

$$(\Delta X_v)_t = \left\{ \frac{V}{[1-(V/c)^2]^{1/2}} \right\} \delta t \tag{5-1}$$

which is derived by differentiating equation (4-1) with respect to time. As shown in Figure 4 this error increases with increasing velocity and becomes very large as the velocity approaches the speed of light.

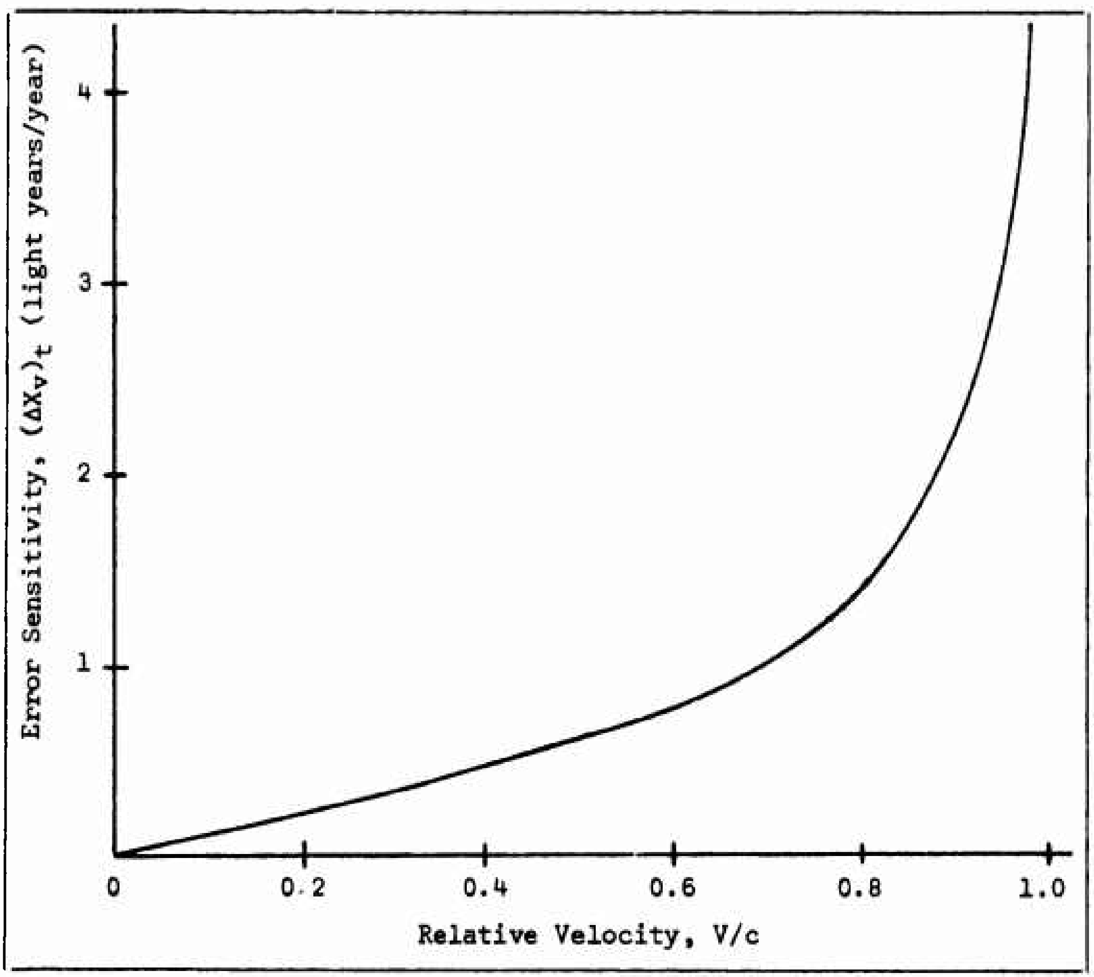

Figure 4. Position Error Resulting from Time Error

Now assume there is an error δV in determining the velocity V. Then the resulting error in position is

$$(\Delta X_V)_V = \left\{ \frac{t}{[1-(V/c)^2]^{3/2}} \right\} \delta V \tag{5-2}$$

which is obtained by differentiating equation (4-1) with respect to velocity. This error will increase with increasing time t and increasing velocity V. As the velocity approaches the speed of light the error becomes very large.

Although this method of determining position can be used for flight planning where all values of velocity and time are assumed to contain no error, it would be unreliable for measuring position during actual flights unless the actual flight is for short time periods at relatively low velocities. Therefore, this method should only be used during flight planning or during the initial phase of an actual flight when both the time and velocity are relatively small.

The spacecraft velocity in constant velocity flight is determined from

$$V = c \left\{ \frac{\cos\theta' \pm \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \right\} \tag{4-6}$$

with the negative sign used when $\cos\theta' > V/c$ and the positive sign when $\cos\theta' < V/c$. It is assumed that there is no error in the inertial wavelength λ. Assume there is an error $\delta\theta'$ in measuring the apparent position angle. Then the resulting error in velocity is

$$(\Delta V)_{\theta'} = c \left\{ \frac{\sin\theta' \left[\cos^2\theta' - \left(\frac{\lambda'}{\lambda}\right)^2\right]}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2} \right.$$

$$\left. \pm \frac{\sin\theta'\cos\theta' \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta' - 1\right]}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2 \left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}} \right\} \delta\theta' \qquad (5\text{-}3)$$

The error sensitivity $(\Delta V)_{\theta'}/\delta\theta'$ is plotted in Figure 5 which shows that the resulting error in spacecraft velocity is minimized when the velocity is determined by measuring the wavelength shift of either the sun or the destination star.

Now assume there is an error $\delta\lambda'$ in measuring the apparent wavelength. The resulting error in spacecraft velocity is

$$(\Delta V)_{\lambda'}$$

$$= c\left\{ \frac{\pm\left[\cos^4\theta' - \cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2(\cos^2\theta'+1)\right] - 2\left(\frac{\lambda'}{\lambda}\right)\cos\theta'\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\lambda\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}} \right\}\delta\lambda'$$

(5-4)

which is obtained by differentiating equation (4-35) with respect to λ'. This error sensitivity $(\Delta V)_{\lambda'}/\delta\lambda'$ is plotted in Figure 6 which shows that the error in velocity due to errors in measuring the wavelength λ' is also minimized when the wavelength measurements are made from either the sun or the destination star.

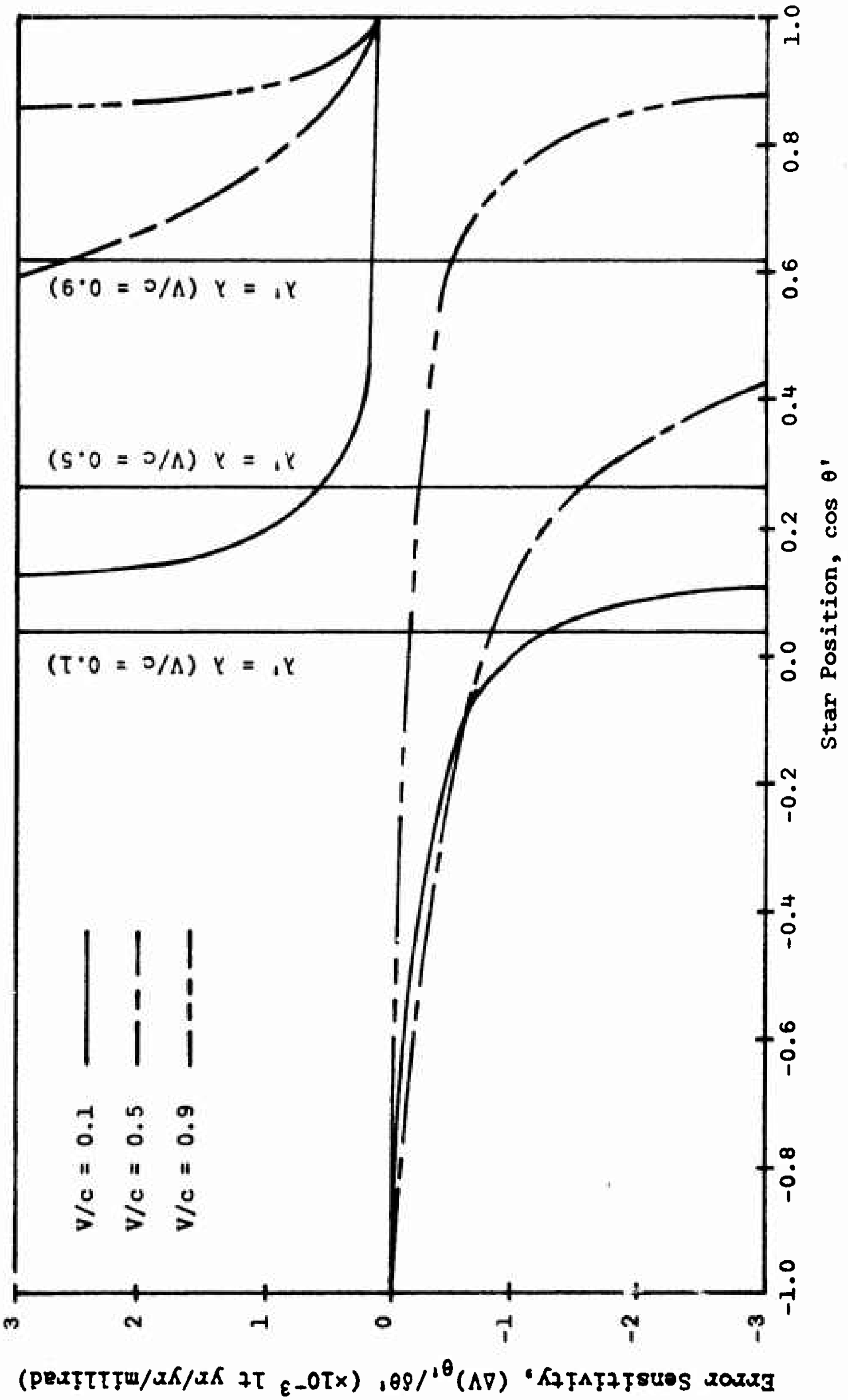

Figure 5. Velocity Error Resulting from Position Angle Error

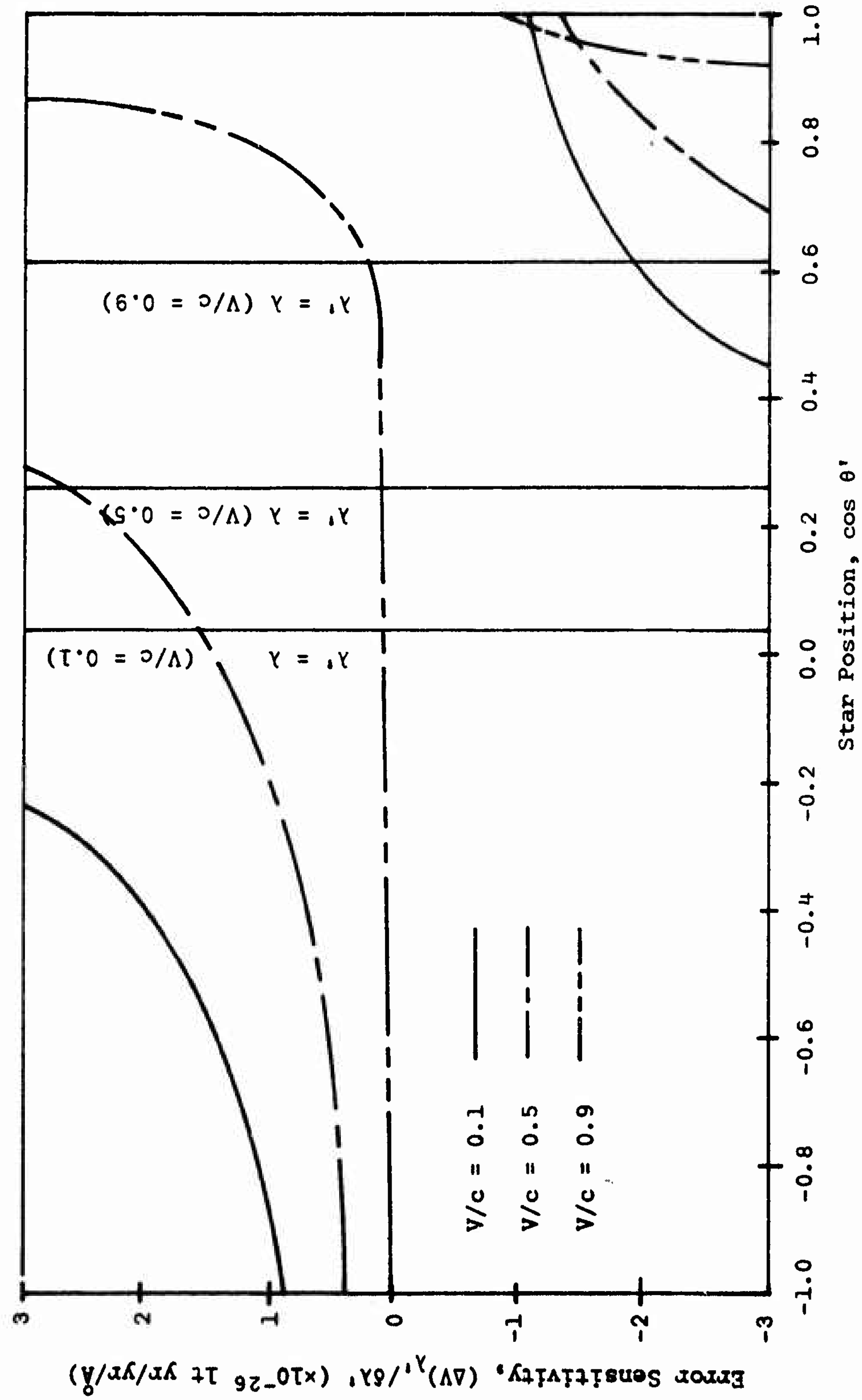

Figure 6. Velocity Error Resulting from Wavelength Error

From this analysis the velocity of the spacecraft is most accurately determined by measuring the apparent wavelength λ' and position angle θ' from either the destination star or the sun. This is fortuitous as the most complete and accurate data available on stars is the data for the sun. Other stars may be used but the reliability decreases and in the case of stars which apparently do not shift wavelength the error becomes very large. The magnitude of the error caused by wavelength measurement errors is larger than that for position angle measurement errors and therefore higher accuracy is required in both the instruments and measurement procedures used to determine the wavelength of light emitted from a star.

The inertial position of the spacecraft in constant velocity flight is determined from

$$X_V = X_S - \frac{Y_S\ (\cos\theta' - V/c)}{\sin\theta'\ [1-(V/c)^2]^{1/2}} \tag{4-19}$$

by measuring the velocity V and the apparent position angle θ'. It is assumed that the position of the reference star (X_S, Y_S) is known. In this analysis it is assumed that the velocity has been measured using the previously developed method. Therefore, the only parameter of interest in this error analysis is the error $\delta\theta'$ in measuring the apparent position angle. The resulting error in position is

$$(\Delta X_V)_{\theta'} = \left\{ \frac{-Y_S\ [1-(V/c)\cos\theta']}{\sin^2\theta'\ [1-(V/c)^2]^{1/2}} \right\} \delta\theta' \tag{5-5}$$

This error sensitivity $(\Delta X_V)_{\theta'}/\delta\theta'$ is plotted in Figure 7 which shows that the error is minimized when the apparent position angle θ' is measured from stars which do not appear to shift in wavelength. The error in position increases as the spacecraft velocity increases. From equation (5-5) it is also apparent that the off-axis distance to the star Y_S should be as small as possible. In addition neither the sun nor the destination star can be used to determine position which is also readily apparent from Figure 2. They are both located on the X axis and no triangulation method exists for determining the position of the spacecraft. This analysis shows that the inertial position of the spacecraft should be determined by measuring the position angle θ' from stars nearest the X axis which do not appear to shift wavelength.

Constant Acceleration Error Analysis

During constant acceleration flight the inertial position of the spacecraft is computed from

$$X_V = (c^2/g)\,[\cosh(gt/c)-1] \qquad (3\text{-}5)$$

where the acceleration g and time t are the measured parameters. Assume there is an error δt in measuring the time. The resulting error in position is

$$(\Delta X_V)_t = \{c \sinh(gt/c)\}\ \delta t \qquad (5\text{-}6)$$

Since the acceleration g and the speed of light c are both constants, this error will be zero initially and increase as the time increases approaching very large values for long flights.

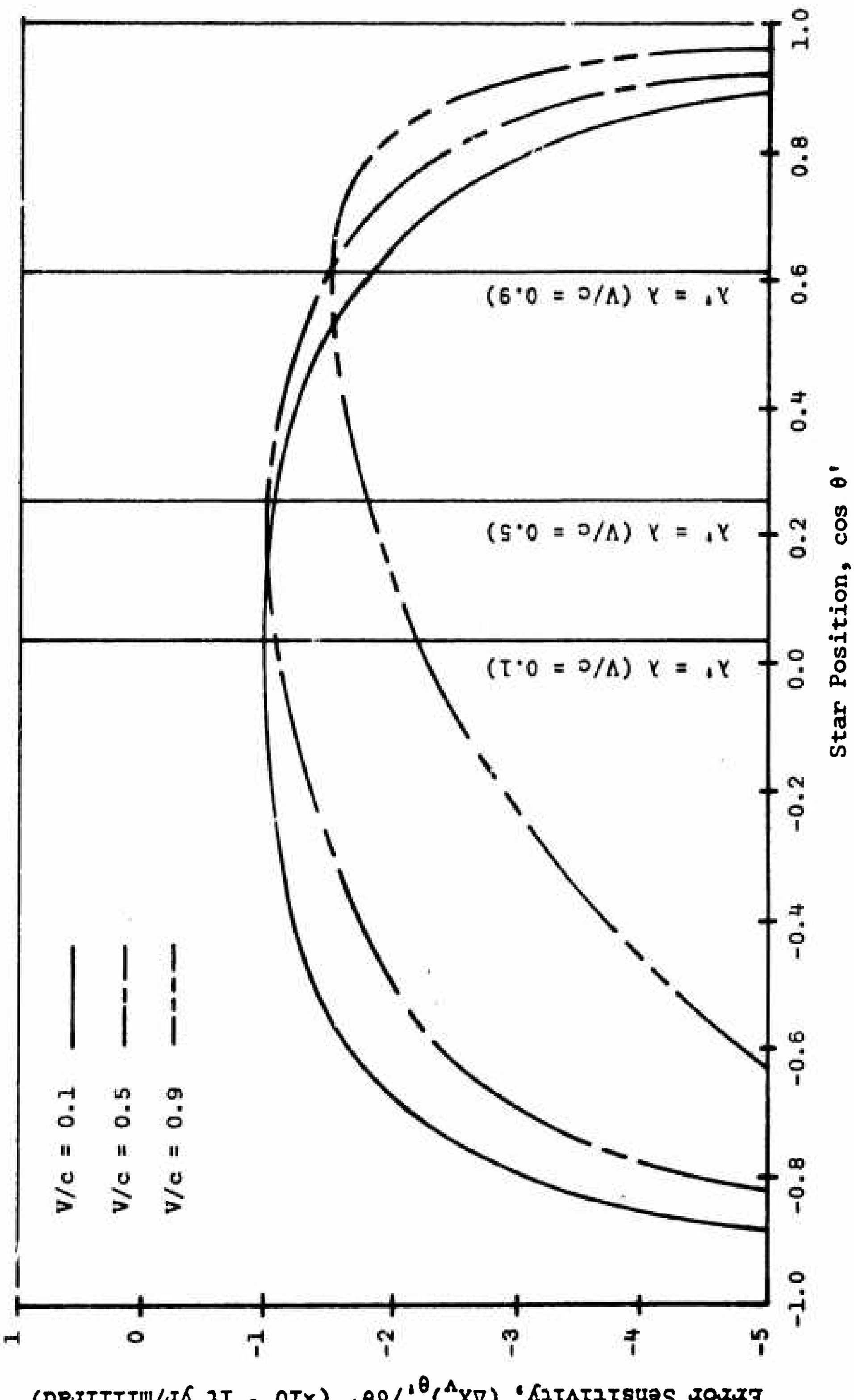

Figure 7. Position Error Resulting from Position Angle Error

Now assume there is an error δg in measuring the acceleration. Then the resulting error in position is

$$(\Delta X_V)_g = \left\{\left(\frac{c}{g}\right)\left[t \sinh\left(\frac{gt}{c}\right) - \frac{c}{g}\cosh\left(\frac{gt}{c}\right) - \frac{c}{g}\right]\right\}\delta g \qquad (5\text{-}7)$$

The error sensitivity $(\Delta X_V)_g/\delta g$ is plotted in Figure 8 which shows that the error decreases initially but begins to increase as time increases and eventually approaches very large values. This analysis indicates that the method for computing inertial spacecraft position is acceptable for flight planning where there are no measurement errors, however it is not acceptable for long duration flights where the error would become very large.

The velocity of the spacecraft can be computed from

$$V = c \tanh(gt/c) \qquad (3\text{-}7)$$

where the acceleration g and time t are again the measured parameters. Assuming a time measurement error δt, the resulting error in velocity is

$$(\Delta V)_t = \left\{\frac{g}{\cosh^2(gt/c)}\right\}\delta t \qquad (5\text{-}8)$$

Assuming an accelerometer error δg in measuring the acceleration, then the error in velocity is

$$(\Delta V)_g = \left\{\frac{t}{\cosh^2(gt/c)}\right\}\delta g \qquad (5\text{-}9)$$

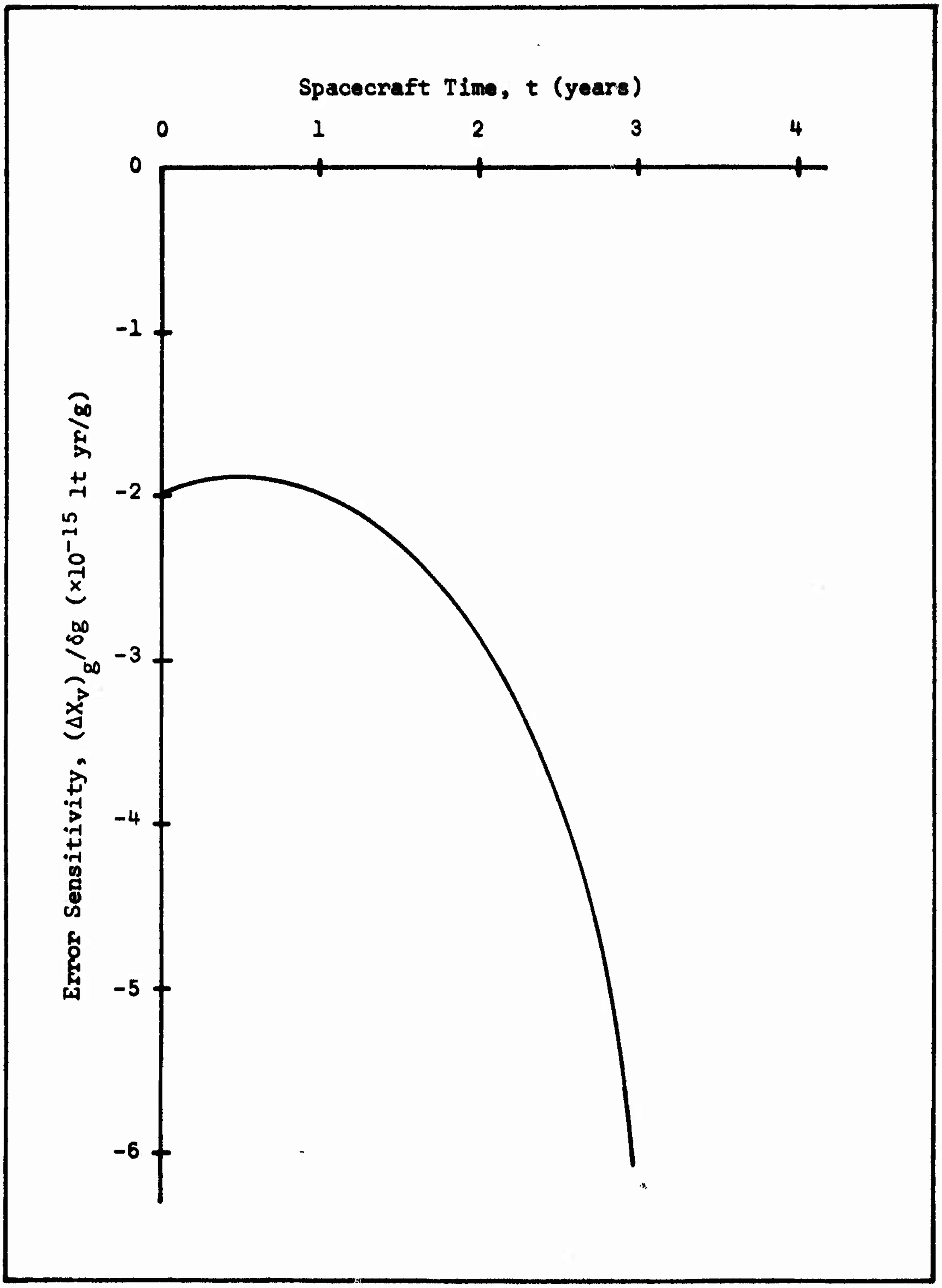

Figure 8. Position Error Resulting from Acceleration Error

The error sensitivities for equations (5-8) and (5-9) are plotted in Figure 9.

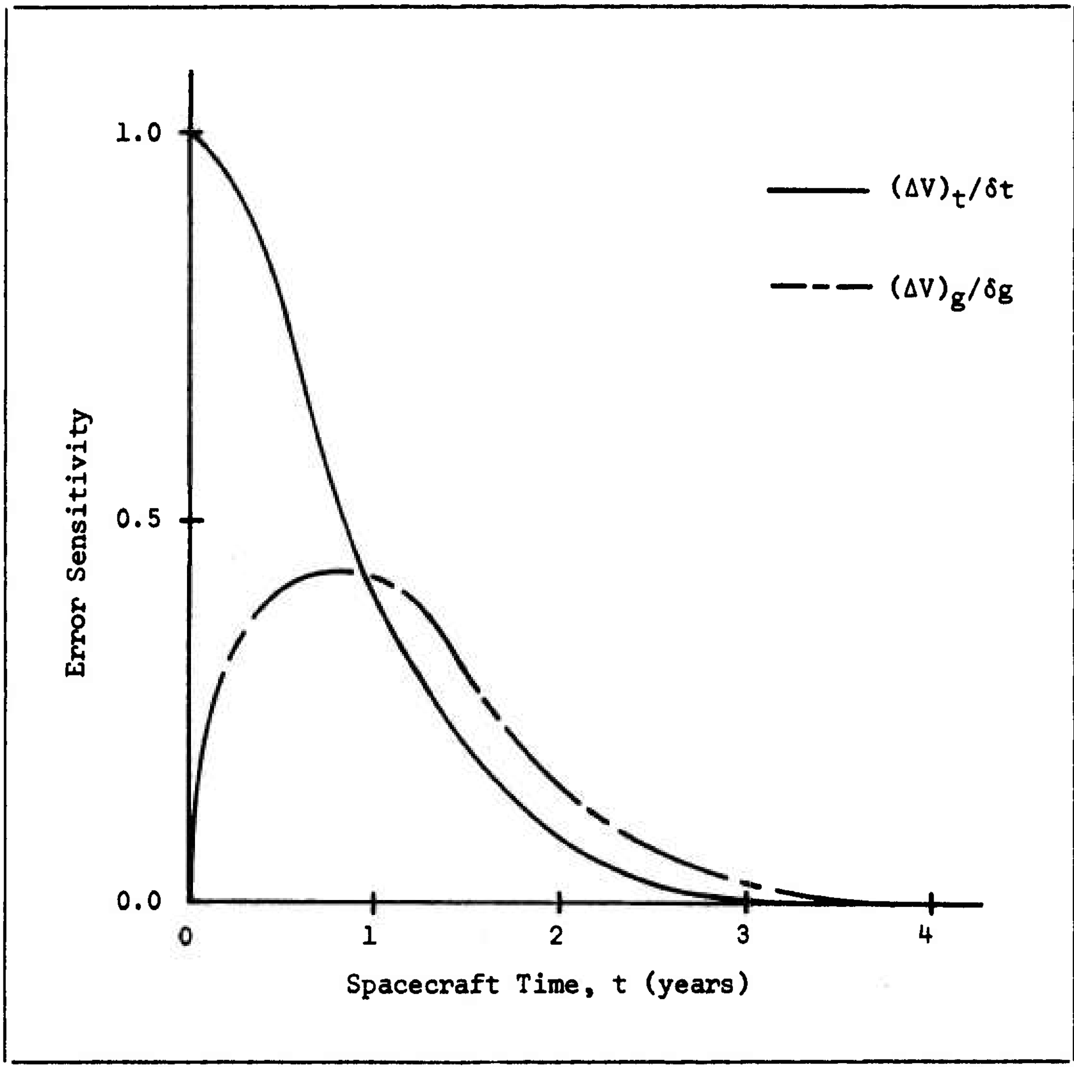

Figure 9. Velocity Error in Constant Acceleration Flight

The error in velocity due to errors in measuring time is a maximum at t = 0 and then decreases as time increases. The error in velocity due to errors in measuring the acceleration increases from zero initially to a maximum at approximately t = 0.7 years and then decreases with further increases in time. The analysis shows that this would be

a reliable method of determining spacecraft velocity if the instrument errors are small. Since current atomic clocks (Ref 6:460) and accelerometers are extremely accurate and reliable, this technique for determining spacecraft velocity can be used during both flight planning and the actual flight.

The velocity measuring technique for constant acceleration is identical to that used for constant velocity flight and therefore the error analysis is the same as that performed in the constant velocity section of this chapter. However, it must be noted that there will be a time lag from the time when the velocity measurement is made equal to the computing time required and the spacecraft will always have a velocity greater than the measured velocity in constant acceleration flight. Therefore, it would be desirable to develop a predictor program which would account for the compilation lag time and provide both the measured and predicted spacecraft velocity at any instant of time.

The position measuring technique for constant acceleration flight is also identical to that used for constant velocity flight and the error analysis would again be identical to that presented in the constant velocity section. In both constant velocity and constant acceleration flight the actual and measured spacecraft position will differ by the computation lag time. Therefore, a predictor program should also be developed for the position measuring system to account for this computation lag time and provide both the measured and predicted spacecraft position at any instant of time.

VI. Conclusions and Recommendations

This study has resulted in the development of expressions for determining the inertial position and velocity of a spacecraft in either constant velocity or constant acceleration flight by measuring the necessary parameters with instruments located on the spacecraft. The error analysis has determined how the measurements should be performed so that the error in determining the inertial position and velocity of the spacecraft is minimized. The results are summarized in the following two sections.

Constant Velocity Flight

For flight planning the inertial position of the spacecraft is determined from

$$X_V = Vt/[1-(V/c)^2]^{1/2} \qquad (4\text{-}1)$$

where the velocity V is specified for the flight and time t is measured on the spacecraft. The error analysis showed that this expression should not be used in actual flight where it is necessary to measure these quantities as the measurement errors make the results unreliable.

During actual flight the velocity of the spacecraft is determined from

$$V = c\left\{\frac{\cos\theta' \pm \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2}\right\} \qquad (4\text{-}6)$$

where c is the speed of light, λ is the inertial wavelength of light emitted from the star, λ' is the measured wavelength of the light from the star, and θ' is the measured position angle to the star. In this equation for velocity, the plus sign is used if $\cos\theta' < V/c$ and the negative sign is used if $\cos\theta' > V/c$. The error in the velocity determination is minimized by measuring the wavelength λ' and position angle θ' from either the sun or the destination star.

The inertial position of the spacecraft is determined from

$$X_V = X_S - \frac{Y_S\,(\cos\theta' - V/c)}{\sin\theta'\,[1-(V/c)^2]^{1/2}} \tag{4-19}$$

where V is the spacecraft velocity, c is the speed of light, θ' is the measured position angle to the star, and X_S and Y_S are the inertial coordinates of the star's position. The error in determining the inertial position X_V is minimized when the position angle θ' is measured from the stars for which there is no apparent wavelength shift to the observer on the spacecraft.

Constant Acceleration Flight

For flight planning the inertial position of the spacecraft is determined from

$$X_V = (c^2/g)\,[\cosh(gt/c)-1] \tag{3-5}$$

where c is the speed of light, g is the specified acceleration for the flight, and t is the time as measured on the spacecraft. This expression should not be used during actual flight as the measurement errors will make the resulting position information unreliable.

The velocity of the spacecraft can be determined from

$$V = c \tanh(gt/c) \tag{3-7}$$

where c is the speed of light, g is the acceleration as measured on the spacecraft, and t is the time as measured on the spacecraft. This expression can be used for either flight planning or during the actual flight where the acceleration and time are measured by instruments on the spacecraft.

In actual flight the velocity of the spacecraft can also be determined from

$$V = c \left\{ \frac{\cos\theta' \pm \left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2} \right\} \tag{4-6}$$

which is identical to the expression developed for constant velocity flight.

The inertial position of the spacecraft during actual flight can be determined from

$$X_V = X_S - \frac{Y_S\,(\cos\theta' - V/c)}{\sin\theta'\,[1-(V/c)^2]^{1/2}} \tag{4-19}$$

which is also identical to the expression developed for constant velocity flight. The measuring techniques and error analysis were identical for flight at either constant velocity or constant acceleration. Therefore, it appears that the visual effects result only from the spacecraft velocity and are independent of any constant acceleration which the spacecraft may undergo.

Recommendations

This was only a conceptual study and therefore there are many areas of investigation remaining in the general relativistic navigation problem. The following are a few possibilities for follow-on studies:

1. Investigate the effect of variations away from the x-axis in the flight path and develop methods of measuring and correcting these variations.

2. Develop the same general type of inertial position and velocity expressions for a spacecraft undergoing constant thrust.

3. Complete the navigation system of the spacecraft by designing a closed loop navigation system including the guidance and control section which would keep the spacecraft on course.

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Appendix A

Derivation of Error Expressions

The error analysis used in this study is based on the standard Taylor's Series expansion method of computing the error in a function of the form $N = f(u_1, u_2, u_3, \cdots, u_n)$ due to an error in one of the variables of δu_n. Assuming the individual errors $\delta u_1, \delta u_2, \cdots, \delta u_n$ are relatively small the squares, products, and higher order terms are neglected and the error in N due to an error in the variable u_n is (Ref 12:9)

$$(\Delta N)_{u_n} = \left[\frac{\partial N}{\partial u_n}\right] \delta u_n \tag{A-1}$$

This form is the basis for developing the error expressions in chapter V and each error expression is formed by taking the appropriate partial derivatives. The remainder of this appendix consists of the differentiation of the equations presented in chapter V to form the appropriate error expressions.

Constant Velocity

$$X_v = \frac{Vt}{[1-(V/c)^2]^{1/2}} \tag{4-1}$$

Differentiation with respect to time yields

$$\frac{\partial X_v}{\partial t} = \frac{V}{[1-(V/c)^2]^{1/2}} \tag{A-2}$$

The resulting error in position due to an error in time is

$$(\Delta X_V)_t = \left\{ \frac{V}{[1-(V/c)^2]^{1/2}} \right\} \delta t \tag{5-1}$$

Differentiation of equation (4-1) with respect to velocity yields

$$\frac{\partial X_V}{\partial V} = \frac{t}{[1-(V/c)^2]^{1/2}} - (1/2)\,\frac{(Vt)(-\,2V/c^2)}{[1-(V/c)^2]^{3/2}} \tag{A-3}$$

which reduces to

$$\frac{\partial X_V}{\partial V} = \frac{t}{[1-(V/c)^2]^{3/2}} \tag{A-4}$$

The error in position due to an error in velocity is

$$(\Delta X_V)_V = \left\{ \frac{t}{[1-(V/c)^2]^{3/2}} \right\} \delta V \tag{5-2}$$

$$V = c \left\{ \frac{\cos\,\theta' \pm (\lambda'/\lambda)[(\lambda'/\lambda)^2 - \sin^2\theta']^{1/2}}{\cos^2\theta' + (\lambda'/\lambda)^2} \right\} \tag{4-6}$$

Differentiation of equation (4-6) with respect to position angle θ' yields

$$\frac{\partial\left(\frac{V}{c}\right)}{\partial\theta'} = \frac{-\sin\theta'\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right] + 2\cos^2\theta'\sin\theta'}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2}$$

$$\pm\frac{\left(\frac{\lambda'}{\lambda}\right)(-\sin\theta'\cos\theta')\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right] + 2\sin\theta'\cos\theta'\left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]}{\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]^{1/2}}$$

(A-5)

Equation (A-5) simplifies to

$$\frac{\partial\left(\frac{V}{c}\right)}{\partial\theta'} = \frac{\sin\theta'\left[\cos^2\theta'-\left(\frac{\lambda'}{\lambda}\right)^2\right]}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2} \pm \frac{\sin\theta'\cos\theta'\left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta' - 1\right]}{\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]^{1/2}}$$

(A-6)

Therefore the error in velocity due to an error in measuring the position angle θ' is

$$(\Delta V)_{\theta'} = c\left\{\frac{\sin\theta'\left[\cos^2\theta'-\left(\frac{\lambda'}{\lambda}\right)^2\right]}{\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right]^2} \pm \frac{\sin\theta'\cos\theta'\left(\frac{\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'-1\right]}{\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]^{1/2}}\right\}\delta\theta'$$

(5-3)

Differentiation of equation (4-6) with respect to apparent wavelength λ' yields

$$\frac{\partial\left(\frac{V}{c}\right)}{\partial\lambda'} = \frac{\pm\left(\frac{1}{2}\right)\left(\frac{4\lambda'^3}{\lambda^4} + \frac{2\lambda'}{\lambda^2}\cos^2\theta' - \frac{2\lambda'}{\lambda}\right)\left[\left(\frac{\lambda'}{\lambda}\right)^4 + \left(\frac{\lambda'}{\lambda}\right)^2\cos^2\theta' - \left(\frac{\lambda'}{\lambda}\right)^2\right]^{-1/2}}{\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2}$$

$$- \frac{\left(\frac{2\lambda'}{\lambda^2}\right)\left\{\cos\theta' \pm \left[\left(\frac{\lambda'}{\lambda}\right)^4 + \left(\frac{\lambda'}{\lambda}\right)^2\cos^2\theta' - \left(\frac{\lambda'}{\lambda}\right)^2\right]^{1/2}\right\}}{\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2} \qquad \text{(A-7)}$$

Simplifying and combining terms

$$\frac{\partial\left(\frac{V}{c}\right)}{\partial\lambda'} = \frac{\pm\left[\frac{\lambda'}{\lambda^2}\cos^4\theta' - \frac{\lambda'}{\lambda^2}\cos^2\theta' + \frac{\lambda'^3}{\lambda^4}\cos^2\theta' + \frac{\lambda'^3}{\lambda^4}\right]}{\left(\frac{\lambda'}{\lambda}\right)\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}$$

$$- \frac{2\,\frac{\lambda'^2}{\lambda^3}\cos\theta'\left[\left(\frac{\lambda'}{\lambda}\right)^2 + \cos^2\theta' - 1\right]^{1/2}}{\left(\frac{\lambda'}{\lambda}\right)\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}} \qquad \text{(A-8)}$$

Further combining results in

$$\frac{\partial\left(\frac{V}{c}\right)}{\partial\lambda'} = \frac{\pm\left[\cos^4\theta' - \cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2(\cos^2\theta' + 1)\right] - 2\left(\frac{\lambda'}{\lambda}\right)\cos\theta'\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}{\lambda\left[\cos^2\theta' + \left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2 - \sin^2\theta'\right]^{1/2}}$$

(A-9)

The error in velocity due to an error in measuring the apparent wavelength λ' is

$$(\Delta V)_{\lambda'} = c\left\{\frac{\left[\cos^4\theta'-\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2(\cos^2\theta'+1)\right]-2\left(\frac{\lambda'}{\lambda}\right)\cos\theta'\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]^{1/2}}{\lambda\left[\cos^2\theta'+\left(\frac{\lambda'}{\lambda}\right)^2\right]^2\left[\left(\frac{\lambda'}{\lambda}\right)^2-\sin^2\theta'\right]^{1/2}}\right\}\delta\lambda' \tag{5-4}$$

$$X_V = X_S - \frac{Y_S\ (\cos\theta' - V/c)}{\sin\theta'\ [1-(V/c)^2]^{1/2}} \tag{4-19}$$

Differentiation with respect to the apparent position angle θ' yields

$$\frac{\partial X_V}{\partial\theta'} = \frac{Y_S}{[1-(V/c)^2]^{1/2}}\left[\frac{-\sin^2\theta' - \cos\theta'\ (\cos\theta' - V/c)}{\sin^2\theta'}\right] \tag{A-10}$$

which reduces to

$$\frac{\partial X_V}{\partial\theta'} = \frac{-\ Y_S\ [1-(V/c)\ \cos\theta']}{\sin^2\theta'\ [1-(V/c)^2]^{1/2}} \tag{A-11}$$

The error in position due to an error in measuring the apparent position angle θ' is

$$(\Delta X_V)_{\theta'} = \left\{\frac{-\ Y_S\ [1-(V/c)\ \cos\theta']}{\sin\theta'\ [1-(V/c)^2]^{1/2}}\right\}\delta\theta' \tag{5-5}$$

Constant Acceleration

$$X_v = (c^2/g)[\cosh(gt/c)-1] \tag{3-5}$$

Differentiation with respect to time yields

$$\frac{\partial X_v}{\partial t} = (c^2/g)\ (g/c)\ \sinh(gt/c) \tag{A-12}$$

The position error due to an error in measuring time is

$$(\Delta X_v)_t = \{c \sinh(gt/c)\}\ \delta t \tag{5-6}$$

Differentiation of equation (3-5) with respect to acceleration yields

$$\frac{\partial X_v}{\partial g} = -(c^2/g^2)[\cosh(gt/c)-1] + (c^2/g)(t/c)\ \sinh(gt/c) \tag{A-13}$$

$$\frac{\partial X_v}{\partial g} = (c/g)[t \sinh(gt/c) - (c/g)\cosh(gt/c) - c/g] \tag{A-14}$$

The position error due to an error in measuring the acceleration is

$$(\Delta X_v)_g = \{(c/g)[t \sinh(gt/c) - (c/g)\cosh(gt/c) - c/g]\}\ \delta g \tag{5-7}$$

$$V = c \tanh(gt/c) \tag{3-7}$$

Differentiation with respect to time results in

$$\frac{\partial V}{\partial t} = c\ (g/c)\ \mathrm{sech}^2(gt/c) \tag{A-15}$$

The error in velocity due to an error in measuring time is

$$(\Delta V)_t = \left\{ \frac{g}{\cosh^2(gt/c)} \right\} \delta t \tag{5-8}$$

Differentiation of equation (3-7) with respect to acceleration yields

$$\frac{\partial V}{\partial g} = c\ (t/c)\ \mathrm{sech}^2(gt/c) \tag{A-16}$$

The error in velocity due to an error in measuring acceleration is

$$(\Delta V)_g = \left\{ \frac{t}{\cosh^2(gt/c)} \right\} \delta g \tag{5-9}$$

Vita

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